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ORIGINAL PAPER

Treatise on Winkler Modulus of Subgrade Reaction and Its Estimation for Improved Soil–Structure Interaction Analysis

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Abstract The paper pertains to the development of a new relationship expressing the Winkler modulus of subgrade reaction as a function of elastic parameters such as the modulus of elasticity and Poisson's ratio of the foundation soil, and the relative rigidity of the foundation soil and the beam resting over the same. It ensures that the maximum values of beam deflection and bending moment computed by using both the theory of elastic continuum and lumped parameter modeling are either identical or very close to each other. In this respect the developed expression can be construed to be superior to those proposed by Biot and Vesic, which respectively predict correct values for only one of the quantities (either maximum bending moment or maximum deflection). In addition, the proposed model for subgrade modulus is applicable to multiple load conditions as well unlike the other two approaches as reported.

Keywords Beams on elastic foundation · Elastic continuum · Young's modulus · Subgrade modulus · Finite difference method · Numerical modeling

1 Introduction

In recent years Winkler model in spite of its limitations (Selvadurai 1979) is being increasingly used by civil engineers in analyzing soil structure interaction problems static (Daloglu and Vallabhan 2000) as well as dynamic (Boulanger et al. 1999; Allotey and El Nagger 2008; Prendergast and Gravin 2016). Use of such models can be justified where the main objective is the analysis of the foundation beam and not of the foundation soil bed.

Objection to the use of Winkler model is also due to the fact that the modulus of subgrade reaction used in the model and as determined by plate load test conducted in the field does not use any fundamental property of soil. To bridge these gap seminal efforts were made quite early by Biot (1937) and subsequently by other investigators like Vesic (1961), Brown (1973); excellent discussions on these methods have been made by several investigators (Selvadurai 1979; Scott 1981; Bowels 2001). As the works of Biot (1937) and Vesic (1961) are widely referred and still

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being used (Sall et al. 2013) these are looked into in some more details.

Biot (1937) and Vesic (1961) proposed expressions relating the modulus of subgrade reaction of foundation soils as a function of the elastic parameters of the foundation soil and the relative rigidity of the foundation soil and the footing resting over it and matched the maximum bending moment and the maximum deflection respectively. Both these approaches to some extent blunt the objection to the use of modulus of subgrade reaction expressing those as a function of the fundamental elastic parameters of the soil and relative rigidity of soil and the footing in finding the response of beams on elastic foundation. However, none of the above two expressions correctly predicts the maximum values of both the beam deflection as well as the bending moment simultaneously. Use of Vesic's model does not ensure prediction of the maximum bending moment developed on the footing treated as a beam on elastic foundation and Biot's model on the other hand does not predict the maximum deflection, which match correspondingly with those by theory of elasticity solutions. Chandrasekaran (2001) reported that none of the two models (Biot's and Vesic's) is able to predict accurately either the maximum bending moment or the deflection for multiple load conditions. It is necessary to study the above aspect in some detail and to improve the models so that predictions with regard to the maximum bending moment and maximum deflection match with the theory of elasticity solutions.

Recently renewed efforts have been made in this direction by Klar et al. (2004) and also Iskander and Gabr (2009) in analyzing Soil-Pipeline interaction due to tunneling. They adopted Winkler springs and Vesic's expression to estimate the modulus of subgrade reaction and found that the equation may not be adequate for such analysis.

The inadequacies of the existing lumped parameter models led several investigators to suggest the use of now proven numerical methods of analysis like fem for analyzing elastic half space problems. But Winkler foundation model due to its simplicity is widely used by the practicing engineers and finding more attention in tackling reinforced soil beds (Shukla and Chandra 1994; Madhav and Poorooshasb 1989; Yin 2000; Albet and Kovacs 2003; Dey 2009) and soil–structure interaction problems under both static and dynamic conditions as reported earlier. Due to the reasons as explained above it is not yet the time to discard the use of Winkler based models in geotechnical engineering research and practice as over the years it has acquitted itself quite admirably in solving such interaction problems.

Therefore, in this study, an effort has been made to provide a better estimate of k as a function of fundamental material properties like E and μ, which would not only provide better estimates of both maximum bending moment and deflection simultaneously, but will also be applicable to multiple load conditions. Thus, the new proposed model is an effort to bridge the gap between the Single parameter lumped parameter model based on Winkler hypothesis and elastic theory approaches. This is achieved by minimizing the squared error between the solutions from the two approaches. The use of the new model for k will also enhance the applicability of those computer software, which had been developed earlier in the fifties and sixties of the last century based on modulus of subgrade reaction and are still frequently used. In addition, in many new models for reinforced foundation beds that are being developed using lumped parameter models, the rheological model parameters can be found as a function of the modulus of elasticity, Poisson's ratio and viscoelastic coefficient in a similar manner as developed in this paper, leading to their greater acceptability. In that sense the developed paper is a small step towards development of newer models combining the flexibility of lumped parameter models and theoretical soundness of continuum mechanics based models. However, it should be kept in mind that some of the deficiencies of the basic model with respect to the lack of continuity beyond the loaded area still persist; the proposed equation may not be the panacea that will give good results for all types of soil–structure interaction problems and it may be necessary to investigate the same.

1.1 Existing Models for Modulus of Subgrade Reaction

For analyzing a flexible foundation for its structural design using the Winkler foundation model it is essential to evaluate the model parameter namely the coefficient of subgrade reaction, k defined as: $k = -p/\Delta$, where a foundation of width b is subjected to a load per unit area of p and the corresponding settlement is Δ. The unit of k is kN/m^3. The value of k is not constant for a soil. It depends on several factors like the length and width of the foundation and on the depth of the foundation as well. A comprehensive study of the parameters affecting the co-efficient of subgrade reaction has been presented by Terzaghi (1955). As per this study the coefficient of subgrade reaction decreases with the width of the foundation. In the field load tests may be conducted on square plates measuring 0.3 m × 0.3 m and the values of k can be calculated. For larger foundations of size bxl, the values of subgrade modulus $k_{(bxl)}$ can be related to the subgrade modulus of the smaller test plate (size 0.3 m × 0.3 m) $k_{0.3}$ using standard expressions as proposed by Terzaghi. Subgrade modulus increases with depth as the foundation settlement is dependent on the modulus of elasticity of soil beneath the foundation, which is a function of the spatial location of the collected sample. It is necessary to plan adequate site investigation and laboratory testing to accurately calibrate the model used so as to ensure that the results obtained are realistic. Several expressions are available for modulus of subgrade reaction accounting for the flexural rigidity of the foundation to which the spring is attached whereas the others specify the spring constants without consideration to geometry or flexibility. The primary issue addressed in this paper is the specification of the stiffness parameter that correctly reflects the response of the soil-beam system with reasonable accuracy. The stiffness parameter, known as the coefficient of modulus of subgrade reaction for static problems is a difficult parameter to specify as it typically varies with loading scheme, geometry of the foundation and the type of subgrade material. Prendergast and Gavin (2016) compared the performance of five different subgrade reaction models that were typically developed for the application to static problems, for use in small-scale dynamic modeling of a pile-soil system. Two most well known and frequently expressions proposed by Biot (1937) and Vesic (1961) correlating the modulus of subgrade reaction (k) as a function of the elastic parameters of the soil and the relative rigidity of the soil are chosen for further exploration. These are:

Biot's Model: Biot (1937) presented a solution for the problem of an infinite beam with a conventional load, resting on a 3-D elastic soil continuum. He correlated the continuum elastic theory and the Winkler model by equating the maximum moments in the infinite beam and developed an empirical relation for the coefficient of subgrade reaction, k_s as shown in Eq. (1a).

$$k_s = \frac{0.95E_s}{(1 - \nu_s^2)}\left[\frac{B^4E_s}{(1 - \nu_s^2)EI}\right]^{0.108} \tag{1a}$$

where E_s is the Young's modulus of the soil elements, B is the width of the foundation element, EI is the flexural rigidity of the foundation element and ν_s is the Poisson's ratio of the foundation soil.

Vesic's Model: Adopting a similar method Vesic (1961) derived an equation for by matching the maximum displacements of an infinite beam as shown in Eq. (2a).

$$k_s = \frac{0.65E_s}{(1 - \nu_s^2)}\left[\frac{B^4E_s}{EI}\right]^{1/12} \tag{2a}$$

Which is of similar form as that of Eq. (1a).The above two expressions are useful in evaluating the modulus of subgrade reaction K (= k x b) from triaxial test conducted under appropriate drainage condition on undisturbed soil samples collected from the field located below the foundation level.

Other relations as proposed by Brown (1973) and Vlasov and Leontiev (Scott 1981) are not discussed here as those are not used very often.

Prendergast and Gavin (2016) noted that the above two most widely used methods for foundation analysis due to Biot's and Vesic's produced values, which differs by about 27%.

1.2 The Developed Methodology

1.2.1 Objective Statement

The objective of the developed methodology is explained with reference to a typical combined footing resting on the surface of a homogeneous, semi-infinite, isotropic and elastic foundation soil medium as shown in Fig. 1. The modulus of elasticity of the footing is E_b.

The length, width and depth of the footing are L, B and d respectively. Thus the moment of inertia (I) of the footing section is equal to $\frac{Bd^3}{12}$. Several transverse point loads, P_i, and moment M_i act at different locations of the footing. The unit weight of soil media is γ. The elastic modulus and Poisson's ratio are taken

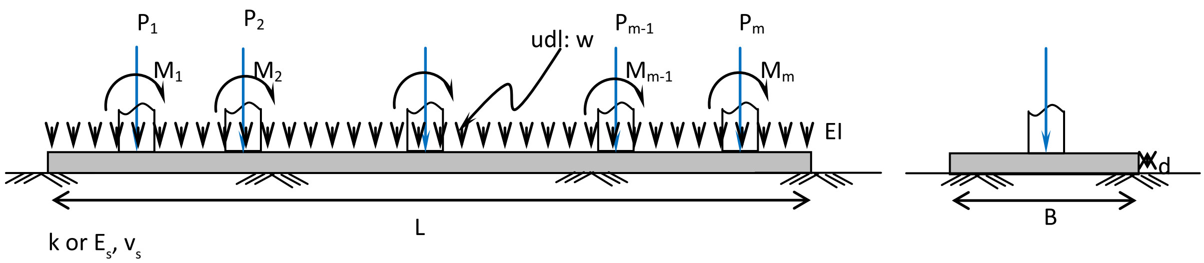


Fig. 1 Definition sketch

as E_s, ν_s respectively; the corresponding subgrade modulus of soil is designated as k. The flexural response of the footing is found out by treating it as a beam on elastic foundation using elastic continuum method and by using the concept of modulus of subgrade reaction as well.

The objective of the paper is to develop new relationships expressing the modulus of subgrade reaction as a function of the elastic parameters of the soil and the relative rigidity of the soil and the structure (footing), such that use of both the above approaches results in values of maximum deflection and bending moments that are either identical or very close to each other matching excellently with theory of elasticity solutions.

1.3 Proposed Models for Estimating Modulus of Subgrade Reaction

As reported none of the expressions for modulus of subgrade reaction [Eqs. (1a) and (2a)] is able to predict simultaneously both the maximum deflection and the bending moment correctly due to the very nature of the matching the Winkler solution and elastic continuum solution equating either the maximum moment (as in Eq. 1a) or the maximum deflection (as in Eq. 2a). As such, it is necessary to formulate the problem so as to achieve both the objectives simultaneously minimizing the prediction error for both the quantities. This can be done by casting the problem as one of optimization.

To meet the laid down objectives, two models are suggested in this study, which are similar to the original form of the Eqs. (1a) and (2a). The proposed models are obtained by introducing two parameters α and β into the existing expressions as follows.

Model 1: Following Biot's expression the chosen model is:

$$k_1 = \frac{\alpha E_s}{(1-\nu_s^2)} \left[\frac{B^4 E_s}{(1-\nu_s^2)EI}\right]^{\beta} \tag{1b}$$

Model 2: The suggested model is similar to Vesic's expression:

$$k_2 = \frac{\alpha E_s}{(1-\nu_s^2)} \left[\frac{B^4 E_s}{EI}\right]^{\beta} \tag{2b}$$

1.4 Optimization Formulation

1.4.1 Design Vector

The parameters α and β appearing in Eqs. (1b) and (2b) need to be determined to estimate the value of the Winkler modulus as a function of the soil and beam properties. Thus, the design vector **D** for the proposed parameter identification problem is:

$$\mathbf{D} = (\alpha, \beta)^T$$

1.4.2 Objective Function

In order to match both the values of the predicted and measured values of the deflection and bending moment we use the well known method of least square. Thus, the objective function is the sum of the squared differences between the predicted and the actual values of deflection and bending moment:

If $M_{i,m}$ and $Y_{i,m}$ are respectively the maximum bending moment and the maximum deflection predicted by using the modulus of subgrade reaction approach and M_m and Y_m are the corresponding values obtained by using theory of elasticity approach then the objective function (Error function) can be expressed as follows.

$$F_i(\mathbf{D}) = (M_m - M_{i1,m})^2 + S_{f_i}(Y_m - Y_{i,m})^2 \tag{3}$$

where $M_{i,m}$ and $Y_{i,m}$ are respectively the maximum bending moment and the maximum deflection predicted by using the modulus of subgrade reaction approach, M_m and Y_m are the corresponding values obtained by using theory of elasticity approach, S_{fi} is a scaling factor and i refers to the model used [Eqs. (1b) or (2b)]. The minimization of the above objective function $F_i(\mathbf{D})$ with respect to α and β ensures that the bending moment and deflection will match from the two approaches irrespective of the choice of model (i = 1, 2).

Because of the difference in the order of magnitude of moment and deflection a suitable scaling factor S_{f1} needs to be introduced in the objective function in order to reduce the number of iterations to converge (Fox 1971). Following Morgenstern and Price (1967) the scaling factor is chosen as,

$$S_{fi} = \frac{\left(\frac{\partial(M_m - M_{i,m})}{\partial k_1}\right)^2 + \left(\frac{\partial(M_m - M_{i,m})}{\partial k_2}\right)^2}{\left(\frac{\partial(Y_m - Y_{i,m})}{\partial k_1}\right)^2 + \left(\frac{\partial(Y_m - Y_{i,m})}{\partial k_2}\right)^2} \tag{4}$$

The partial derivatives appearing in the above expression are computed by using central difference scheme of finite difference.

1.4.3 Minimization of the Objective Function

In the absence of any constraints imposed on the design variable, minimization of the objective function (Eq. 3) is carried out with respect to the design parameters α and β by using Powell's (1964) conjugate direction method of multidimensional search and quadratic fit for unidirectional search. The method chosen is expected to have quadratic convergence. The method is well known and available in standard text books (Fox 1971).

1.4.4 Computation of Deflection and Bending Moment

This section presents the methods for computing the deflection and bending moment values appearing in Eqs. (3) and (4) based on the lumped parameter model and the elastic continuum model. The governing equations and solution methods for both the approaches are presented.

1.5 Governing Differential Equations

1.5.1 Winkler Model

Referring to figure below (Fig. 2), the footing treated as a beam is divided into n equal segments of length Δx with (n + 1) nodes. The soil response is idealized by reaction from discrete mechanical elements like linearly elastic springs.

Thus the reactions R_j from foundation soil support at each node can be determined as follows:

$$\begin{aligned} R_1 &= \frac{1}{2} k y_1 \Delta x \\ &- \\ &- \\ R_j &= k y_j \Delta x \\ &- \\ R_{N+1} &= \frac{1}{2} k y_{N+1} \Delta x \end{aligned} \tag{5}$$

where k is modulus of subgrade reaction, y_j is deflection at node j and Δx is footing segment length.

The governing differential equations to represent the flexural response of the combined footing idealized as a beamBending Equation:

$$\frac{d^2y}{dx^2} = -\frac{M_x}{EI} \tag{6}$$

Using central difference scheme the above differential equation can be written in a finite difference form as follows:

$$y_{i+1} - 2y_i + y_{i-1} = -\frac{M_i}{EI}\Delta x^2 \tag{7}$$

where M_i is the bending moment at ith point that can be determined as a function of R_j, P_j, uniformly distributed load w and applied moment. Therefore, (n − 1) equations can be established for i = 2, 3, 4,…, n. To avoid the use of fictitious points, end points are not used to write the equation. The two other equations necessary for solving the (n + 1) unknown come from the two equilibrium equations, namely the overall force and moment equilibrium equations as follows.

Force equilibrium ($\sum F_v = 0$)

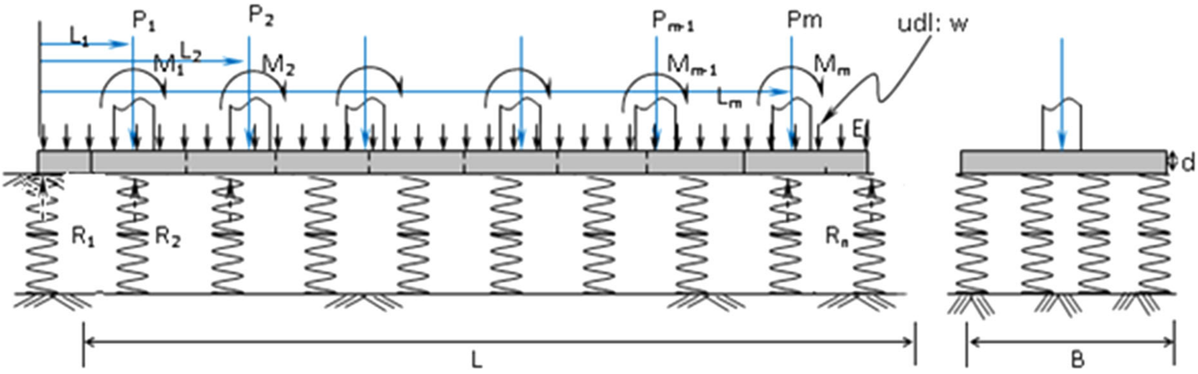


Fig. 2 Definition sketch of lumped parameter model

$$\sum_{j=1}^{n+1} R_j - \sum_{j=1}^{m} P_j - wL = 0 \tag{8}$$

Moment equilibrium $(\sum M_i = 0)$

$$\sum_{j=2}^{n+1} R_j \times (j-1) \times \Delta x - \sum_{j=1}^{m} P_j \times L_j - \frac{w \times L^2}{2} - \sum M_{applied} = 0 \tag{9}$$

where R_j is the soil reaction which is a function of nodal deflections from Eq. (5), P_j are acting column loads and L_j is the distance of loads from left most node number 1. Thus, the (n + 1) unknown deflections y's can be determined by solving these (n + 1) equations using Gauss elimination technique.

1.5.2 Elastic Continuum Model

Referring to figure (Fig. 3) above, the footing is divided into n equal segments of length Δx. The soil response is idealized as linearly elastic continuum with elastic parameters E_s and v. It is assumed that the footing beam is supported on elastic soil continuum by reactions R_j at each node from foundation soil. Hence footing beam is also applying equal and opposite reactions load at each node on the foundation soil (Fig. 4). Using this analogy we can establish a relationship between foundation soil reaction R_j and deflections y_j.

From theory of elasticity, the vertical displacements at the surface of an semi-infinite elastic medium acted upon by a concentrated load R_j are given by the expression:

$$y_j = \frac{R_j(1-\mu^2)}{\pi r E_S} \tag{10}$$

This equation breaks down at r = 0, however if it is assumed that load R_j is applied to soil through a square plate of dimension BxB then the deflection of the surface immediately below the load R_j is given by (Smith and Pole 1980)

$$y_o = \frac{3R_j(1-\mu^2)}{\pi B E_S} \tag{11}$$

Thus,

Deflection at node 1 due to $R_1 = \frac{3R_1(1-\mu^2)}{\pi BE_S}$

Deflection at node 1 due to $R_2 = \frac{R_2(1-\mu^2)}{\pi(BE_S)(\Delta x/B)}$

Deflection at node 1 due to $R_3 = \frac{R_3(1-\mu^2)}{\pi(BE_S)(2\Delta x/B)}$

Similarly the contribution from the other nodes can also be written.

Deflection at node 1 due to $R_{n+1} = \frac{R_{n+1}(1-\mu^2)}{\pi(BE_S)(n\Delta x/B)}$

So, total deflection at node 1

$$y_1 = \frac{3R_1(1-\mu^2)}{\pi BE_S} + \frac{R_2(1-\mu^2)}{\pi(BE_S)(\Delta x/B)} + \cdots + \frac{R_{n+1}(1-\mu^2)}{\pi(BE_S)(n\Delta x/B)}$$

Similarly, total deflection at node 2

$$y_2 = \frac{R_1(1-\mu^2)}{\pi(BE_S)(\Delta x/B)} \frac{3R_2(1-\mu^2)}{\pi BE_S} + \cdots + \frac{R_{n+1}(1-\mu^2)}{\pi(BE_S)((n-1)\Delta x/B)}$$

and other nodes are,

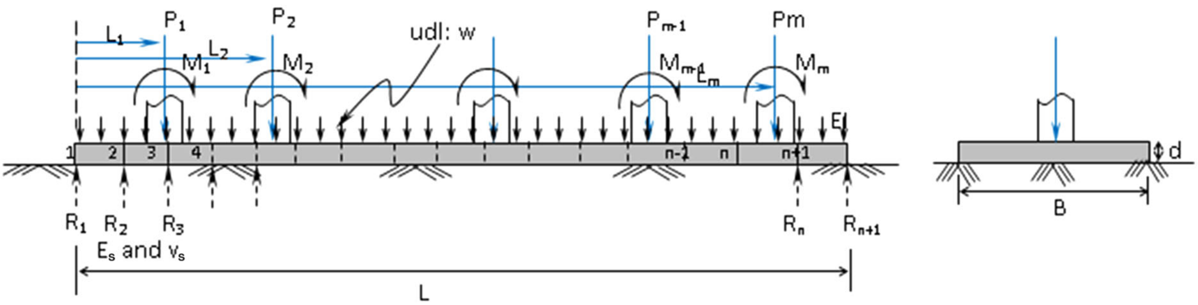


Fig. 3 Definition sketch of elastic continuum model

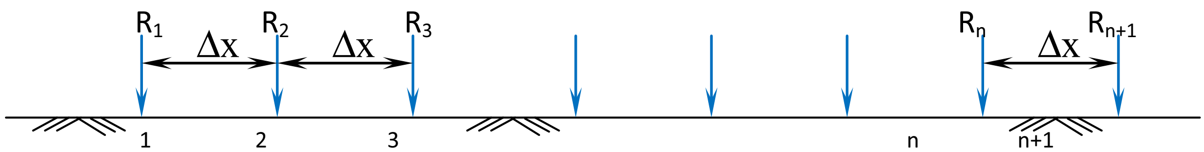


Fig. 4 Reaction Forces on the Soil (elastic continuum)

$$y_{n+1} = \frac{R_1(1-\mu^2)}{\pi(BE_S)(n\Delta x/B)}\frac{R_2(1-\mu^2)}{\pi(BE_S)((n-1)\Delta x/B)} + \cdots + \frac{3R_{n+1}(1-\mu^2)}{\pi BE_S}$$

The nodal deflections can be written in terms of the nodal reaction forces as follows:

$$\begin{Bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_{n+1} \end{Bmatrix} = \frac{(1-\mu^2)}{\pi BE_s} \times \begin{bmatrix} 3 & B/\Delta x & B/2\Delta x & B/3\Delta x & \cdot & \cdot & \cdot \\ B/\Delta x & 3 & B/\Delta x & B/2\Delta x & & & \\ B/2\Delta x & B/\Delta x & 3 & \cdot & & & \\ B/3\Delta x & B/2\Delta x & \cdot & 3 & \cdot & & \\ \cdot & & & \cdot & \cdot & \cdot & \\ \cdot & & & & \cdot & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot & 3 \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \\ \\ \\ R_{n+1} \end{Bmatrix} \tag{12}$$

The governing differential equations for the flexural response of the footing and its finite difference form (Eq. 7) have already been stated (Eq. 6),

The two other equations necessary for solving the (n + 1) unknown come from the two equilibrium equations: Force equilibrium and Moment equilibrium of the footing under the application of loads are written same as stated earlier and represented by Eqs. (8) and (9).

The unknown deflections y's can be determined by solving these (n + 1) equations using gauss elimination method (Fig. 5).

1.6 Non-dimensional form of Governing Equations

1.6.1 Winkler Model

The equations above are converted into non-dimensional form by making use of the following non-dimensional parameters

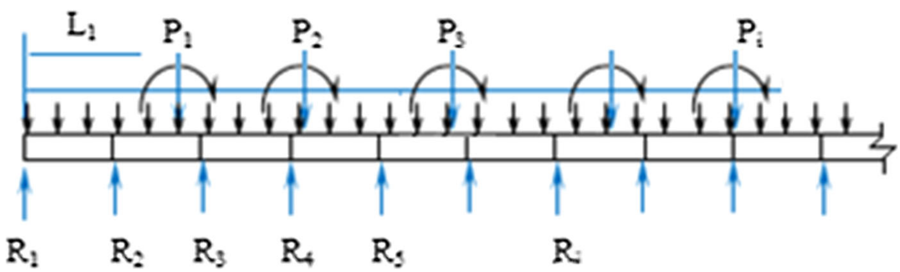


Fig. 5 Reaction forces on the beam

$$Y_i = \frac{y_i}{L},\quad \Delta x' = \frac{x}{L},\quad \lambda = \left(\frac{k}{4EI}\right)^{1/4},$$
$$Q_i = \frac{P_i L^2}{EI},\quad L'_k = \frac{L_k}{L},$$
$$\beta_k = \begin{cases} 0.5 & \text{if } k = 1 \text{ or } n+1 \\ 1.0 & \text{otherwise} \end{cases} \tag{13}$$
$$\sum M'_{applied} = \frac{L}{EI}\sum M_{applied},\quad w' = \frac{wL^3}{EI}$$

Bending equation (finite different form):

$$y_{i+1} - 2y_i + y_{i-1} = -\frac{M_i}{EI}\Delta x^2 \tag{14}$$

Moment at any node i:

$$M_i = \sum_{j=1}^{i-1} \beta_k \times k \times \Delta x^2 \times y_j \times (i-j) + \sum_{j=1}^{m} P_j \times (\Delta x \times (i-1) - L_j) + \frac{w \times ((i-1)\times \Delta x)^2}{2} + \sum M_{applied} \tag{15}$$

M_i as represented by Eq. (15) is put in bending equation (Eq. 14) above; it is then expressed in a non-dimensional form as:

$$4(\lambda L)^4 \times (\Delta x')^4 \sum_{j=1}^{i} \beta_j \times (i-j) \times Y_j + \{Y_{i+1} - 2Y_i + Y_{i-1}\} = -(\Delta x')^2 \sum_{j=1}^{m} Q_j \times \left(\Delta x'(i-1) - L'_j\right) - \frac{w' \times ((i-1)\times \Delta x')^2}{2} - \sum M'_{applied} \tag{16}$$

The force and moment equilibrium equations are as follows:

$$\sum_{j=1}^{n+1} R_j - \sum_{j=1}^{m} P_j - wL = 0 \tag{17}$$

$$\sum_{j=2}^{n+1} R_j * (j-1) * \Delta x = \sum_{j=1}^{m} P_j * L_j + \frac{wL^2}{2} - \sum M_{applied} \tag{18}$$

The above expressions are written in a non-dimensional form as,

$$4(\lambda L)^4 * (\Delta x') \sum_{k=1}^{n+1} \beta_k * Y_k = \sum_{k=1}^{m} Q_k + w' \tag{19}$$

$$4(\lambda L)^4 * (\Delta x')^2 \sum_{k=1}^{n+1} (k-1) * Y_k = \sum_{k=1}^{m} Q_k * (L'_k) + \frac{w'}{2} - \sum M'_{applied} \tag{20}$$

where

$$Y_i = \frac{y_i}{L},\quad \Delta x' = \frac{x}{L},\quad \lambda = \left(\frac{k}{4EI}\right)^{1/4},\quad S_i = \frac{R_i L^2}{EI},$$
$$Q_i = \frac{P_i L^2}{EI},\quad L'_k = \frac{L_k}{L},$$
$$J = \frac{(1-\nu^2)}{\pi},\quad \beta_k = \begin{cases} 0.5 & \text{if } k = 1 \text{ or } n+1 \\ 1.0 & \text{otherwise} \end{cases}$$
$$\sum M'_{applied} = \frac{L}{EI}\sum M_{applied},\quad w' = \frac{wL^3}{EI} \tag{21}$$

1.6.2 *Elastic Continuum Model*

The set of equations (Eq. 12) are converted into non-dimensional form by making use of the non-dimensional parameters as defined earlier:

$$\begin{Bmatrix} Y_1 \\ Y_2 \\ . \\ . \\ Y_{n+1} \end{Bmatrix} = \frac{EkI}{E_s BL^3} \begin{bmatrix} 3 & B/a & B/2a & B/3a & . & . & . \\ B/a & 3 & B/a & B/2a & & & \\ B/2a & B/a & 3 & . & & & \\ B/3a & B/2a & . & 3 & . & & \\ . & & & . & . & . & \\ . & & & & . & . & . \\ . & & & & . & . & 3 \end{bmatrix} \begin{Bmatrix} S_1 \\ S_2 \\ \\ \\ S_{n+1} \end{Bmatrix} \tag{22}$$

Moment at any node i:

$$M_i = \sum_{j=1}^{i-1} R_j \times (i-j) \times \Delta x + \sum_{j=1}^{m} P_j \times (\Delta x \times (i-1) - L_j) + \frac{w \times ((i-1) \times \Delta x)^2}{2} + \sum M_{applied} \tag{23}$$

Putting this M_i in bending equation above and writing the equation in a non-dimensional form,

$$(\Delta x')^3 \sum_{j=1}^{i-1} S_j * (i-j) + \{Y_{i+1} - 2Y_i + Y_{i-1}\} = -(\Delta x')^2 \sum_{j=1}^{m} Q_j * \left(\Delta x'(i-1) - L_j'\right) - \frac{w' \times ((i-1) \times \Delta x')^2}{2} - \sum M'_{applied} \tag{24}$$

Force equilibrium

$$\sum_{i=1}^{n+1} S_i = \sum_{i=1}^{m} Q_i + w' \tag{25}$$

Moment equilibrium

$$\sum_{i=2}^{n+1} S_i * (i-1) * \Delta x' = \sum_{i=1}^{m} Q_i * L_i' + \frac{w'}{2} - \sum M'_{applied} \tag{26}$$

1.7 Summary of the Detailed Procedure

A brief account of the detailed procedure as developed is presented below:

1. Read: a. The geometry of the footing (length, L; breadth, B; and depth, d), soil parameters (k, E_s and ν) and beam material (E)
   b. The magnitudes of the Vertical loads, P_i and Moments, M_i and their locations.
2. Compute moment of inertia of the beam.
3. For any given problem compute the deflection and moment at different node points along the length of the beam by using theory of elasticity approach (Elastic continuum model) and pick up the corresponding maximum values of deflection and moment.
4. Now compute the deflection and moment at the same node points as chosen in step 3 using the Winkler model and pick up the corresponding maximum values.
5. Construct the objective function as defined by Eqs. (3) and (4) and minimize the same by using optimization module of MATLAB staring from an initial design point $\mathbf{D} = (\alpha, \beta)^T$.
6. The process is continued till the convergence is achieved for a desired accuracy. Otherwise steps 4 to vi are repeated till the convergence is reached.

2 Results and Discussions

A computer program was developed in C language to implement the developed procedure to match theory of elasticity and Winkler model solutions to sort out the shortcomings cropping up with the use of models due to Biot (1937) and Vesic (1961). The developed computer program has been calibrated and validated with reference to several example problems for finding the flexural response of combined footing resting on soil beds and subjected to the multiple loadings as shown.

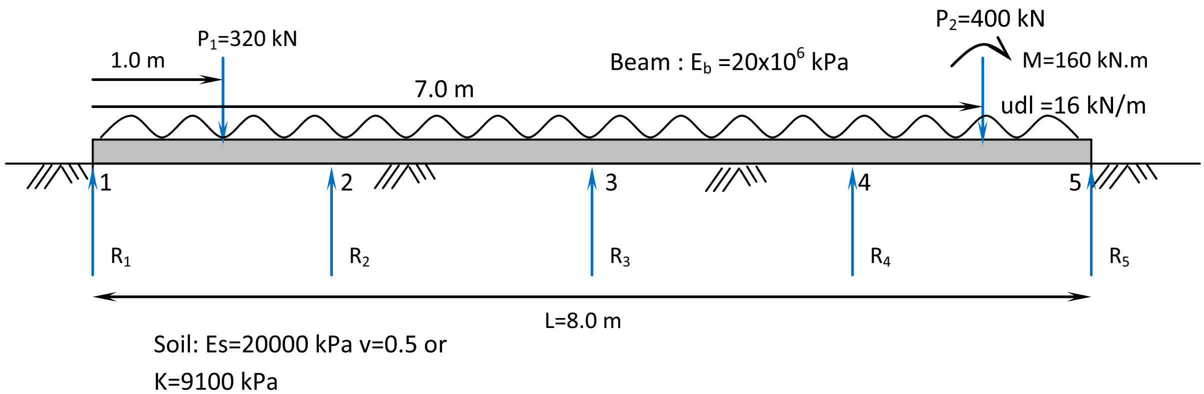


Fig. 6 Definition sketch of example problem 1. (Reproduced with permission from Smith and Pole 1980)

Table 1 Comparison of the present solution with those reported by Smith and Pole (1980)

Node	Deflection (m)[a]		Soil reaction (kN)[a]		Soil reaction (kN)[b]	
	Present study	Smith and Pole (1980)	Present study	Smith and Pole (1980)	Present study	Smith and Pole (1980)
1	− 0.0139	− 0.0138	126.553	125.9	173.970	173.9
2	− 0.0092	− 0.0093	168.227	168.9	137.898	137.7
3	− 0.0073	− 0.0072	132.716	131.2	113.334	113.2
4	− 0.0121	− 0.0122	219.676	222.0	159.757	159.8
5	− 0.0221	− 0.0221	200.828	201.4	263.041	263.0
Sum	–	–	848.000	849.4	847.93	847.6
Error	–	–	0	1.4	0.07	0.4

[a]Modulus of Subgrade Reaction Approach (Winkler Model)

[b]Continuum Mechanics Approach (Linearly elastic)

2.1 Example Problem 1: Validation of the Developed Program

The following benchmark problem of a combined footing subjected to vertical loads and moments for geometric and material properties as shown in Fig. 6 is used to show the correctness of the developed program with respect to the flexural response of the footing. The values of deflection and bending moment at different section of the beam is predicted by using the developed program as a part of the study using both elastic continuum and Winkler model adopting the expressions developed for the purpose. The predicted vales are then compared with the solution provided by Smith and Pole (1980).

In Table 1 the values of the deflection and soil reaction at the designated nodal points as obtained from the present analysis are presented and compared

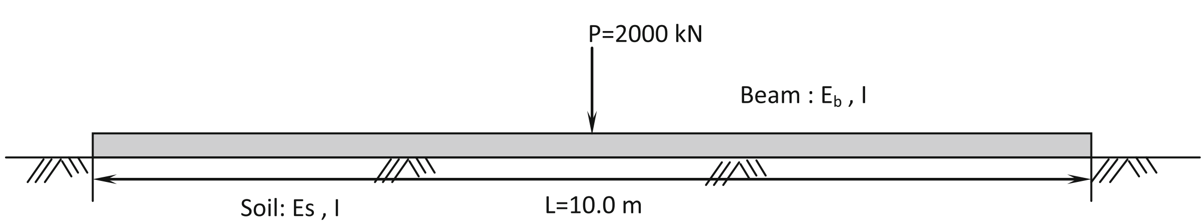


Fig. 7 Definition sketch of example problem 2

with the respective corresponding values reported by Smith and Pole (1980). It is to be noted that the computed deflections using Winkler model is directly compared with those reported by Smith and Pole (1980) whereas the soil reaction values are compared for both the solutions obtained by Winkler and elastic continuum models.

The comparison shows excellent agreement. Summation of all the reactions should be equal to the externally applied load (848.0 kN). Deviation is indicated as an error in the total load in the table. With Winkler model the present solution shows no error whereas in contrast the results obtained by Smith and Pole (1980) shows a small error of 1.4 kN. But for elastic theory, the deviation are negligible small the values being 0.07 and 0.4 only. The moment equilibrium have been identically satisfied. After establishing the correctness of the developed program sensitivity studies are conducted to establish the number of discrete elements in which the footing is to be divided for achieving convergent solution. It is studied with reference to a simpler problem as follows.

2.2 Example Problem 2: Convergence Study

A rectangular beam resting on elastic, homogeneous, isotropic and semi infinite foundation soil is considered as shown in Fig. 7. The soil and footing data are as follows:

Length of the footing L = 10.0 m, Width of the footing B = 1.0 m, Thickness of the footing H = 0.5 m, Modulus of subgrade reaction k = 2000 kPa/m run, Young's modulus of the footing material E_b = 20 GPa, Young's modulus of the foundation soil E_s = 20 MPa, Poisson's ratio of the foundation soil v_s = 0.5, Point load at the midpoint of the beam P = 2000 kN.

Convergence study was made by increasing the number of elements in which the beam has been divided and thereby decreasing the element size. For a typical set of input values of relative stiffness of soil and flexural rigidity of the beam, the effect of number of elements on the normalized mid span beam deflection is shown in Figs. 8 and 9. It is observed that the numerical solution converges when the number of elements exceeds 122 for Winkler model and 80 for elastic continuum model beyond which there is no change in the values of the solution and, as such, these values are subsequently used for further studies. With the help of the above two example problems the correctness of the developed program and convergence of the solutions have been demonstrated. Thereafter some studies are made to check the validity of some of the reported critical observations regarding the relationships proposed by Biot (1937) and Vesic (1961).

2.3 Critical Studies on Biot's (1937) and Vesic's (1961) Solutions

For the centrally loaded beam as shown in Fig. 7, the trend of the variation of deflection along the length of the footing as estimated by using Biot's and Vesic's relations (Eqs. 1a and 2a) for evaluation of modulus of subgrade reaction k in Winkler's model and elastic continuum mechanics approach are shown in Fig. 10.

It is observed that the deflection values as obtained by Winkler model in conjunction with the expressions for modulus of subgrade reaction (Eqs. 1a and 2a) differs significantly from the elastic continuum

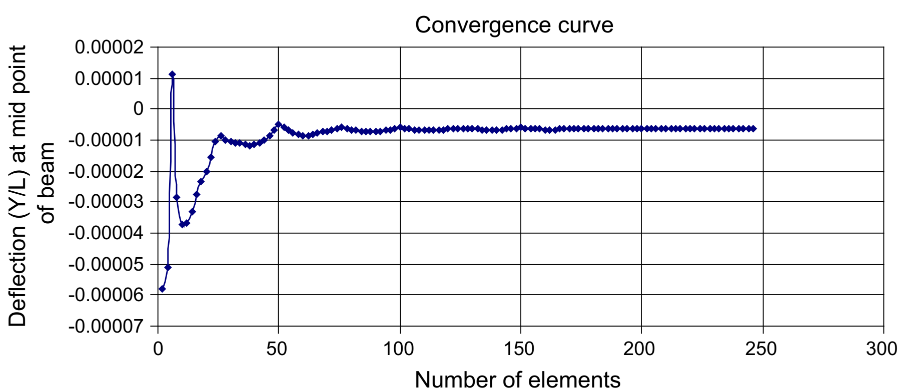
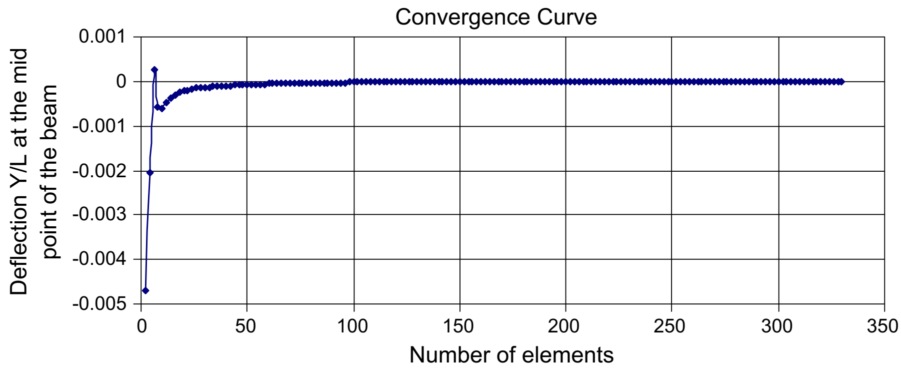


Fig. 8 Convergence curve for determination of optimal number of elements (Winkler Model)

Fig. 9 Determination of optimal number of elements (Elastic Continuum Model)



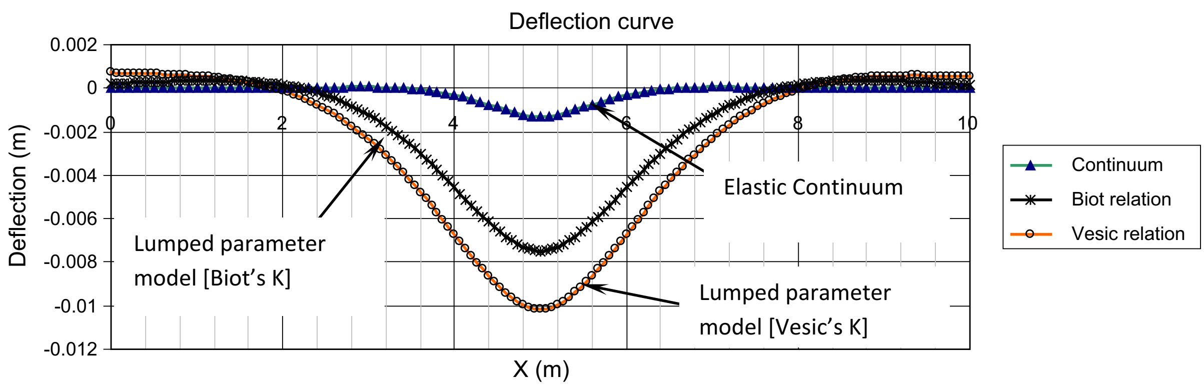


Fig. 10 Variation of beam deflection along its length (Example problem: 2)

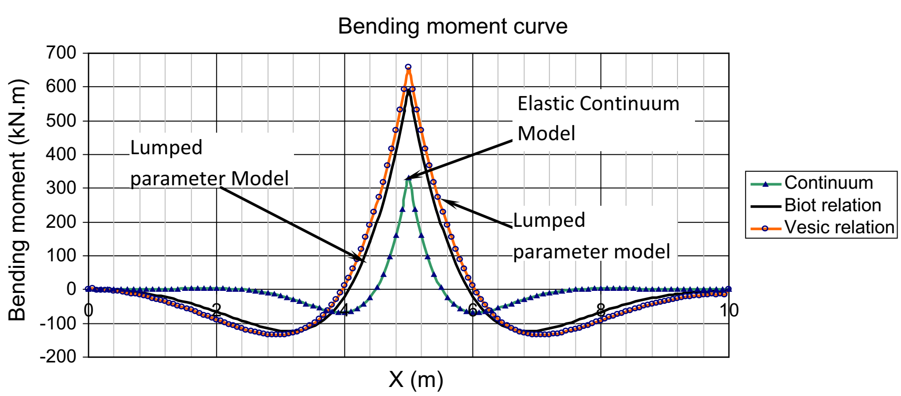


Fig. 11 Variation of bending moment along its length (Example problem: 2)

approach have significant difference, and consideration of the foundation as an elastic continuum results in the least value of the maximum deflection (approximately 1.5 mm) occurring below the point load. When the modulus of subgrade reaction of the foundation is estimated by using Biot's relation, it results in substantial increase in the value of the maximum deflection (approximately 7.5 mm) but the value is smaller than that (approximately 10 mm) when it is computed by using Vesic's relationship. Thus, it is seen that use of Vesic's expression for k results in the maximum value of the maximum deflection contrary to the normal expectation that the maximum deflection of the beam when computed by

Table 2 Comparative error in predictions (Example Problem 2)

Elastic modulus of soil	Elastic continuum approach		Lumped parameter approach using standard Biot's relation		% Error hi deflection	% Error in bending moment	Lumped parameter approach using standard Vesic's relation		% Error hi deflection	% Error in bending moment
E_s (MPa)	Y_m (m)	M_m (kN m)	Y_{1m} (m)	M_{1m} (kN m)	$\frac{Y_m - Y_{1m}}{Y_m}$	$\frac{M_m - M_{1m}}{M_m}$	Y_{2m} (m)	M_{2m} (kN m)	$\frac{Y_m - Y_{2m}}{Y_m}$	$\frac{M_m - M_{2m}}{M_m}$
10	− 0.00779	558.17	− 0.00552	1077.52	29.1399	− 93.044	− 0.0073	1194.67	6.2901	− 114.034
100	− 0.00139	312.47	− 0.00779	558.271	− 460.647	− 78.662	− 0.0106	618.537	− 661.871	− 97.949
200	− 0.00083	262.059	− 0.00434	460.242	− 422.892	− 75.625	− 0.0061	512.383	− 634.94	− 95.5224

using the Vesic's expression for k is likely to be identical to the corresponding continuum mechanics solution. Instead, the value computed by using Biot's expression for k results in a solution that is closer but still quite different from the continuum mechanics solution; but the differences between the values of maximum deflections are significant.

Similarly the variation of bending moment along the length of the footing computed by using the approaches as reported above is presented in Fig. 11.

It is seen from the figure that the predicted value of maximum bending moment from continuum mechanics approach is significantly lower from the same predicted by lumped parameter models (Winkler springs). Here the moment value obtained (approximately 325 kN-m) from continuum mechanics approach is supposed to be matching with that predicted by using k value estimated (600 kN-m approximately) by using Biot's relationship. However, the computed value of bending moment using Vesics's (1961) expressions (650 kN-m approximately) is closer to the value predicted by using Biot's (1937) expression but far from that predicted by continuum mechanics approach.

In Table 2, a quantitative error analysis has been presented. It is observed that when E_s is 10 Mpa, the absolute errors in estimated deflections (when computed by using Vesic and Biot's expressions) with respect to the continuum mechanics solution are about 6 and 29% respectively. For moments it is about 114% and 93% on the conservative side. Thus the errors in the predicted values of moments are very high and cannot be allowed as the design would be very costly. To explore this aspect, further computations were made with higher values of E_s. It is observed that the higher E_s values the corresponding errors are much higher. Thus, from the above study, it can be concluded that for general predictions the expressions as suggested by Biot (1937) and Vesic's (1961) are not sufficient and their use is not proper. The studied example problem reinforces the observation made by Chandrasekaran (2001) that neither Biots expression nor Vesics expression can simultaneously predict the values of maximum deflection and maximum bending moment matching with the corresponding elastic solution.

Therefore, there is a need to explore and develop new expressions for the modulus of subgrade reaction that will be useful for a wider range of values for the elastic parameters.

2.4 Present Approach: Development of New Models for Modulus of Subgrade Reaction

Therefore, for beter predictions an attempt has been made here to develop new expressions similar to those of Biot's and Vesic's. The objective function is so chosen that on minimization both the bending moment and deflection from the two approaches (i.e. the Winkler model and elastic continuum model) match excellenly. The optimized values of the parameter α and β appearing in the expression for modulus of subgrade reaction in Model 1 and Model 2 and the corresponding values of the objective function for different E_s values different from those used earlier are shown in Table 3

The following average values of the parameters, α and β, as computed are chosen for Model 1 and Model 2.

Model 1: α = 10.3194 β = 0.023583

Model 2: α = 10.227876 β = 0.036522

Modulus of subgrade reaction values were computed using the averaged values of α and β in Model 1 (Eq. 1b) and Model 2 (Eq. 2b); the deflection and bending moment adopting the developed approach are presented in Table 4. These values of α and β were checked for several other loading configurations in prediction of bending moment and deflection.

Finally the bending moment and deflection curves are presented in Figs. 12 and 13 for visual inspection and comparison.

The comparison shows that with Model 1 better predictions are obtained matching with the continuum mechanics solution (Fig. 12). However, for bending moments the predictions made using both Model 1 and Model 2 and the elastic continuum solutions are closer to each other with minimal and insignificant error (Fig. 13).

The analysis has been carried out for several soil and beam parameter values and loading configuration. The results are presented as follows in Tables 5 and 6:

From the results presented in Tables 5 and 6, it can be seen that the relative error in deflection with the change in E_s values have marginal effect, the error ranging from 0 to 0.13% and so also for moment, which for all practical purpose can be considered to be zero. But, it can be seen that the values of the parameters k_1 and k_2 differs for different E_s values. As the absolute difference in these values are not much (for α it is 0.63 for model 1 and 0.88244 for model 2; for β it is -0.04009 for model 1 and -0.079184 for model 2) average values of these parameters is suggested to be used. When Model 1 is adopted the average value of α is 10.3194 and β is 0.023583 where as in case of Model 2 the average value of α is 10.227876 and β is 0.036522.

Using the above sets of α and β values, another problem with different loading configuration has been analyzed and the results are Presented as follows to check if the solutions obtained for a different configuration of loading (Example Problem 3) leads to any error or not. Other geometric and material properties are same as specified in the example problem 2 (Fig. 14).

It is observed that the predicted values of deflection and moment from Model 1 and Model 2 are close to the elastic continuum solution. For E_s equal to 100 MPa the error in deflection is − 0.37313% when Model 1 is used and 2.98507 when Model 2 is used; the error in the value of the moment is − 0.00033% from Model 1 and − 0.16092% from Model 2. It suggests that the values of α and β are not significantly affected by the loading configuration.

Table 3 Optimized Values of the parameters α and β

Elastic modulus of soil E_s (MPa)	Lumped parameter approach using Model 1 (Eq. 1b)		Objective function E_1 (k_1, k_2)	Lumped parameter approach using Model 2 (Eq. 2b)		Objective function E_2 (k_1, k_2)
	α	β		α	β	
10	10.6329	0.011065	0.000004	10.62911	0.010086	1.1088E−05
100	10	0.051155	4.01E−08	9.74667	0.08927	2.1E−09
200	10.325388	0.00853	1.53E−08	10.307849	0.010209	1.54E−08

Table 4 Comparison of the present solution with elastic solution (Example Problem: 2)

Elastic modulus of soil	Elastic continuum approach		Lumped parameter approach using Model 1		% Error in deflection	% Error in bending moment	Lumped parameter approach using Model 2		% Error in deflection	% Error in bending moment
E_s (MPa)	Y_m (m)	M_m (kN m)	Y_{1m} (m)	M_{1m} (kN m)	$\frac{Y_m - Y_{1m}}{Y_m}$	$\frac{M_m - M_{1m}}{M_m}$	Y_{2m} (m)	M_{2m} (kN m)	$\frac{Y_m - Y_{2m}}{Y_m}$	$\frac{M_m - M_{2m}}{M_m}$
10	− 0.00779	558.17	− 0.00778	558.169	0.12837	0.00012	− 0.0078	558.179	− 0.12836	− 0.00175
100	− 0.00139	312.47	− 0.00139	312.7	0	− 0.0736	− 0.00139	312.31	0	0.0512
200	− 0.00083	262.059	− 0.00083	262.058	0.1205	3.82E−7	− 0.00087	262.058	0.3614	7.63E−7

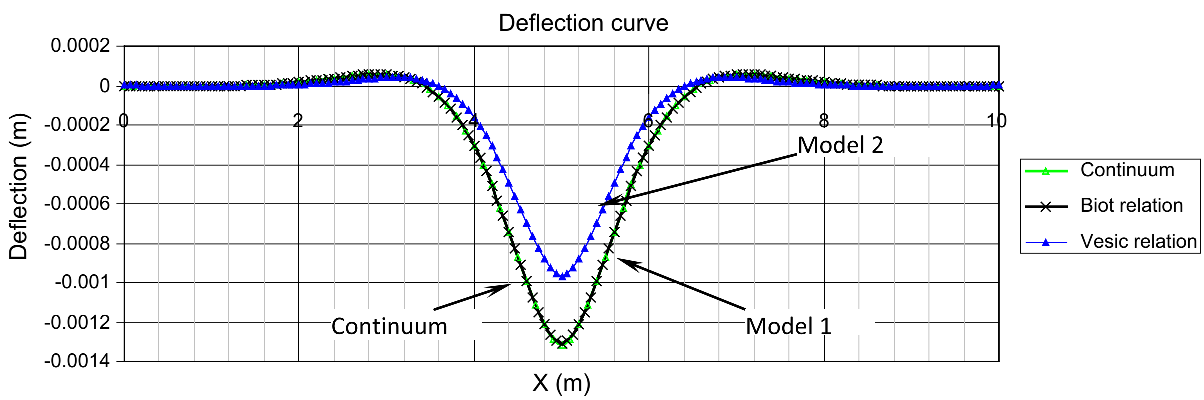


Fig. 12 Variation of beam deflection with length using the optimized parameters (Example problem 2)

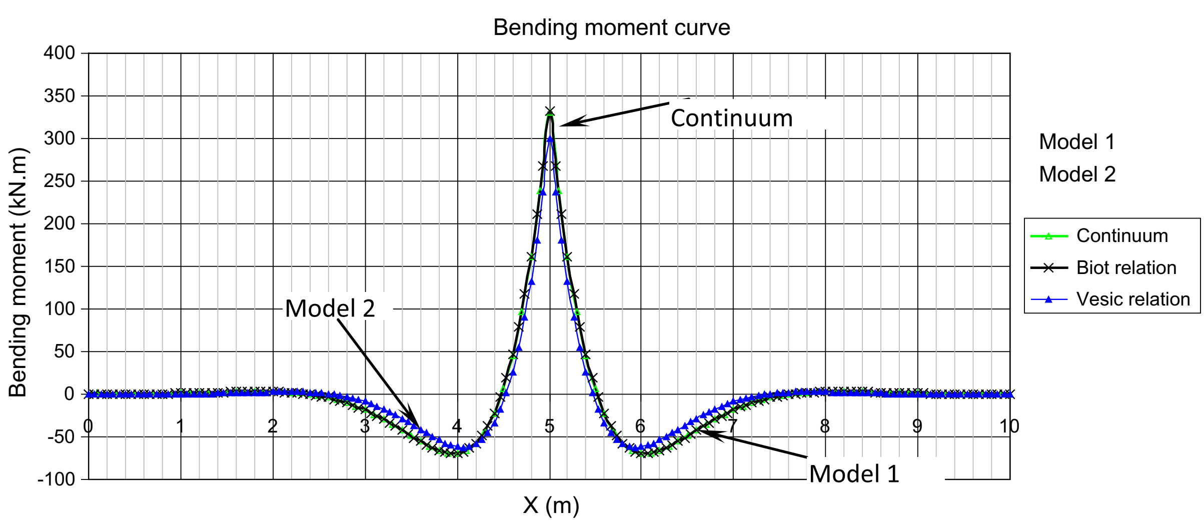


Fig. 13 Variation of bending moment with length using optimized parameters (Example problem: 2)

Now a detailed study is presented as follows to see if instead of using average values of α and β, better predictions could be made using α and β values for different ranges of E_s values.

2.5 Variation of α and β Values with Soil Elastic Modulus

From the above studies, it is observed that the values α and β are not affected significantly by loading

Table 5 Comparison of the solution from elastic continuum approach and Winkler Model

Elastic modulus of soil	Elastic continuum approach		Winkler Model using standard Biot's relation		% Error in deflection	% Error in bending moment	Winkler Model using standard Vesic's relation		% Error in deflection	% Error in bending moment
E_s (MPa)	Y_m (m)	M_m (kN m)	Y_{1m} (m)	M_{1m} (kN m)	$\frac{Y_m - Y_{1m}}{Y_m}$	$\frac{M_m - M_{1m}}{M_m}$	Y_{2m} (m)	M_{2m} (kN m)	$\frac{Y_m - Y_{2m}}{Y_m}$	$\frac{M_m - M_{2m}}{M_m}$
10	− 0.0168	894.014	− 0.169	1773.94	− 905.952	− 98.4242	− 0.2329	2040.74	− 1286.30	− 128.267
100	− 0.00268	609.00	− 0.01682	895.271	− 527.612	− 47.0067	− 0.0241	961.33	− 799.253	− 57.8538
200	− 0.00162	523.51	− 0.00877	794.319	− 441.358	− 51.7294	− 0.01254	847.982	− 674.074	− 61.9800

configuration but varies with soil elastic modulus E_s. To highlight this aspect the variation of modulus of subgrade reaction (k_1 and k_2) with E_s are plotted and presented in Figs. 15 and 16.

An empirical correlation has been established as follows between k_1 and E_s by fitting a quadratic curve as shown in Fig. 16. The value of regression coefficient shows excellent co-relation.

It has already been established that the relationship as given by Model 1 (Eq. 1b) gives better results than the same computed by using Model 2 (Eq. 2b). Now to choose the appropriate values of α and β irrespective of loading condition and modulus of elasticity values the following study is made.

If average values of α and β are taken to be 10.30911 and 0.023559968 respectively the prediction will have an error ranging from − 0.00033% to 2.98507. This range of error can be considered to be negligible. The predicted values are very sensitive to the number of digits after the decimal and, as such, should not be rounded off and be used as given.

Here also the regression co-efficient values in the three zones show excellent co-relations.

From the data presented in Fig. 13, a quadratic curve for α can be fitted,

$$\alpha = 2 \times 10^{-05} E_s^2 - 0.0144 E_s + 10.875$$

$$R^2 = 0.9736$$

Similarly, for k_2, from Fig. 15 three zones were identified to establish three separate curves to check if better predictions could be made. These are:

Zone 1:

$$\beta = 0.0001\ E_s^3 - 0.0039 E_s^2 + 0.0362 E_s - 0.0051$$

$$R^2 = 0.95453$$

Zone 2:

$$\beta = -0.0008\ E_s^2 + 0.0787 E_s - 1.8976$$

$$R^2 = 1$$

Zone 3:

$$\beta = 8.0 \times 10^{-06} E_s^2 - 0.0004 E_s + 0.0145$$

$$R^2 = 1$$

Then the maximum error will lie with in the range − 0.0100265–2.0561001% for zone 1,

Table 6 Comparison of the present solutions (Models 1 and 2) with elastic solution

Elastic modulus of soil	Elastic continuum approach		Lumped parameter approach using Model 1		% Error in deflection	% Error in bending moment	Lumped parameter approach using Model 2		% Error in deflection	% Error in bending moment
E_s (MPa)	Y_m (m)	M_m (kN m)	Y_{1m} (m)	M_{1m} (kN m)	$\frac{Y_m - Y_{1m}}{Y_m}$	$\frac{M_m - M_{1m}}{M_m}$	Y_{1m} (m)	M_{2m} (kN m)	$\frac{Y_m - Y_{2m}}{Y_m}$	$\frac{M_m - M_{2m}}{M_m}$
10	− 0.0168	894.013	− 0.01681	895.182	− 0.07143	− 0.13065	− 0.0168	895.187	− 0.05952	− 0.13126
100	− 0.00268	609.00	− 0.00269	609.006	− 0.37313	− 0.00033	− 0.0026	609.984	2.98507	− 0.16092
200	− 0.00162	523.51	− 0.00269	609.74	− 0.37313	− 0.1215	− 0.0027	609.873	− 0.74626	− 0.14334

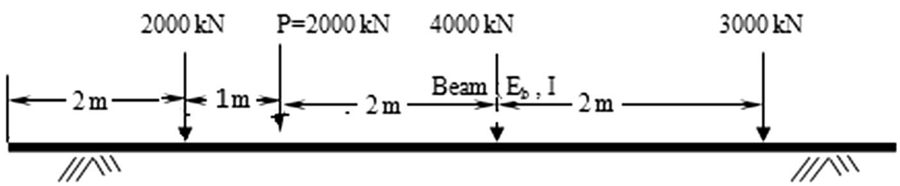


Fig. 14 Example Problem 3: studies on the effect of loading configuration

− 0.01250064–2.530255% for zone 2 and − 0.0150687–3.0001205% for zone 3.

Thus it is seen that by dividing the curve in three zones and predicting the values of β does not result in better predictions. The order of error is similar to the one where average values for these two parameters were chosen. It is observed that for long beams using Model 1, the errors are less than 0.1. 0.6 and 0.5% for long, intermediate and short beams respectively. The same with Model 2 are 0.05, 0.4 and 3.3% respectively. Therefore it can be concluded that the proposed relationship for the modulus of subgrade reaction can be used irrespective of its' length implying a significant improvement in the present state of development.

2.6 Effect of Characteristic Length

It is well established that use of Vesic's (1961) expression works out excellently well for estimating the flexural response of long beams only with an error of the order of 2%. For finite beam Vesic (1967) suggested the use of procedures developed by De Beer, Habel, Ohde and Krsmanovitch. He showed that for intermediate length the use of conventional K-method results in the values of maximum error in deflection, bending moment and the contact pressure is of the order of − 15, 12 and − 20%. However, in this study it has been found that even for long beam the

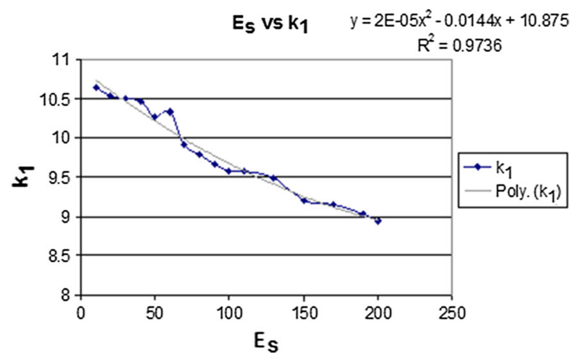


Fig. 15 Variation of K_1 with elastic modulus of foundation soil

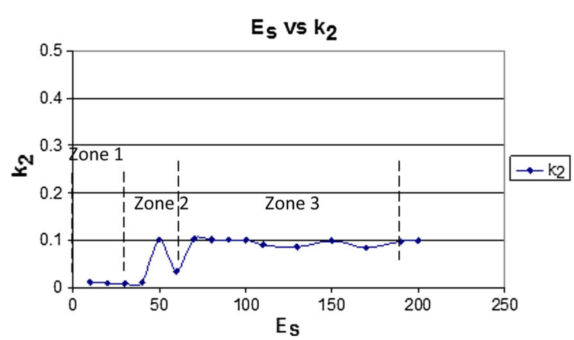


Fig. 16 Variation of k_2 with elastic modulus of foundation soil

error could be very high with increasing value of E_s (Table 7). It is now checked whether the suggested expression as developed (based on the conventional

Table 7 Predictions of deflection and bending moment for long, intermediate and Short Beams using the Present Approach and its comparison with elastic solution

Model	Long beam		Int. beam		Short beam		Elastic conti.		% Error	
	Y_{max}	M_{max}	Y_{max}	M_{max}	Y_{max}	M_{max}	Y_{max}	M_{max}	Y_{max}	M_{max}
1	− 0.00139	312.7					− 0.00139	312.47	0	− 0.0736
2	− 0.00139	312.31					− 0.00139	312.47	0	0.0512
1			− 0.008851	571.499207			− 0.008941	568.272217	0.888	− 0.566
2			− 0.008851	566.23005			− 0.008941	568.272217	1.006	0.35
1					− 0.012695	279.7596	− 0.012768	278.384	0.11	0.49
2					− 0.012502	269.1052	− 0.012768	278.384	0.13	3.33

K-method) in the present study could be extended to beams of long, intermediate and short length.

The example problem that has been presented earlier is solved with varying length meeting the criteria of the beam being long, intermediate and short. The computed values of maximum deflection and moment along with the error visa vis the theory of elasticity solution are presented in Table 7. It can be seen from these two models that for all the types of beams (long, intermediate and short) the computed values of maximum deflection do not differ much from each other but the difference in the predictions of maximum bending moment increase as the characteristics length is decreased. The absolute error for long, intermediate and short beams being 0.39, 5.26 and 10.66 respectively (Table 7). But it is noteworthy that the percentage differences of the present solutions using traditional k-method vary from those from the elastic continuum approach by no more than 0.888 and 3.33% with regard to maximum deflection and bending moment (Table 7).

3 Conclusions

The following conclusions are drawn from the study as reported:

The developed method bridges the gap between the methods based on conventional modulus of subgrade reaction and the elastic continuum. Use of the newly developed expressions (Model 1 and Model 2) for the modulus of subgrade reaction results in excellent predictions of both the maximum values of deflection and bending moment simultaneously unlike the methods proposed by Biot and Vesic, which are capable of predicting either the value of the maximum deflection or the maximum value of bending moment. The respective errors in using the model 1 and model 2 are less than 0.1 and 3.3%. Performance of Model 1 is marginally superior to that of Model 2. The obtained solutions match excellently with hose predicted by elastic continuum method. The predicted values also show that the developed method is applicable for different loading configuration irrespective of the length of the beam (long, intermediate or short). Average values of the model parameters valid for model 1 and model 2 can be used in either of the models effectively for different problems. In this respect the proposed expressions are superior to those presented by Biot and Vesic.

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